MalPath DFE analysis

The simplified model for analysis is the following:

$$\left\{\begin{array}{rcl} x' &=& \Lambda - \mu_x x - \beta m x - \alpha j x, \\ y' &=& \beta m x - \mu_y y - \alpha j y, \\ m' &=& q r \mu_y y - \mu_m m - \beta m x, \\ f' &=& a y(t-\tau) e^{-kj} - \mu_f f, \\ j' &=& b f - \mu_j j. \end{array}\right. \tag{1}$$

Its general Jacobi matrix is

$$J = \begin{pmatrix} -(\mu_x + \beta m + \alpha j) & 0 & -\beta x & 0 & -\alpha x \\ \beta m & -(\mu_y + \alpha j) & \beta x & 0 & -\alpha y \\ -\beta m & r\mu_y & -(\mu_m + \beta x) & 0 & 0 \\ 0 & a e^{-kj} & 0 & -\mu_f & -aky e^{-kj} \\ 0 & 0 & 0 & b & -\mu_j \end{pmatrix}.$$

Theorem 1 *Define*

$$R_0 = \frac{\beta\Lambda(r-1)}{\mu_m \mu_x}.$$

Then the disease-free equilibrium (DFE), namely $(\Lambda/\mu_x, 0, 0, 0, 0)$*, is locally asymptotically stable (LAS) if* $R_0 < 1$.

Proof: At the DFE,

$$J = \begin{pmatrix} -\mu_x & 0 & -\frac{\beta\Lambda}{\mu_x} & 0 & -\frac{\alpha\Lambda}{\mu_x} \\ 0 & -\mu_y & \frac{\beta\Lambda}{\mu_x} & 0 & 0 \\ 0 & r\mu_y & -(\mu_m + \frac{\beta\Lambda}{\mu_x}) & 0 & 0 \\ 0 & a & 0 & -\mu_f & 0 \\ 0 & 0 & 0 & b & -\mu_j \end{pmatrix}.$$

Clearly, three eigenvalues are negative ($\lambda_3 = -\mu_x$, $\lambda_4 = -\mu_j$ and $\lambda_5 = -\mu_f$). The final two eigenvalues are the eigenvalues of the submatrix,

$$J_1 = \begin{pmatrix} -\mu_y & \frac{\beta\Lambda}{\mu_x} \\ r\mu_y & -\mu_m - \frac{\beta\Lambda}{\mu_x} \end{pmatrix}.$$

Since all parameters are strictly positive, the trace of this matrix obviously is negative. The remaining criterion for the system to be LAS is for $\det J_1 > 0$,

$$\Leftrightarrow \mu_m + \frac{\beta\Lambda}{\mu_x} - r\frac{\beta\Lambda}{\mu_x} > 0 \Leftrightarrow 1 > \frac{\beta\Lambda(r-1)}{\mu_m\mu_x}.$$

This completes the proof.